Exercise 21

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k},$$
$$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 1$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^1 \langle \sin x(t), \cos y(t), x(t)z(t) \rangle \cdot \frac{d}{dt}\langle t^3, -t^2, t\rangle\,dt \\
&= \int_0^1 \langle \sin t^3, \cos(-t^2), (t^3)(t) \rangle \cdot \langle 3t^2, -2t, 1\rangle\,dt \\
&= \int_0^1 \langle \sin t^3, \cos t^2, t^4 \rangle \cdot \langle 3t^2, -2t, 1\rangle\,dt \\
&= \int_0^1 [(\sin t^3)(3t^2) + (\cos t^2)(-2t) + t^4(1)]\,dt \\
&= \int_0^1 (\sin t^3)(3t^2)\,dt - \int_0^1 (\cos t^2)(2t)\,dt + \int_0^1 t^4\,dt.
\end{aligned}$$

Make the following substitutions in the first two integrals.

$$\begin{aligned}
u &= t^3 & \qquad v &= t^2 \\
du &= 3t^2\,dt & \qquad dv &= 2t\,dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{0^3}^{1^3} (\sin u)\,du - \int_{0^2}^{1^2} (\cos v)\,dv + \int_0^1 t^4\,dt \\
&= \int_0^1 \sin u\,du - \int_0^1 \cos v\,dv + \int_0^1 t^4\,dt \\
&= (-\cos u)\Big|_0^1 - (\sin v)\Big|_0^1 + \left(\frac{t^5}{5}\right)\Big|_0^1 \\
&= (-\cos 1 + \cos 0) - (\sin 1 - \sin 0) + \left(\frac{1}{5}\right) \\
&= -\cos 1 - \sin 1 + \frac{6}{5}.
\end{aligned}$$